

Generation of orbital angular momenta of photons by transformation media

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To enable an optical beam to carry orbital angular momenta (OAM) it is necessary to form a helical wavefront on the beam. Here, we propose a scheme to generate photonic OAM by using a metamaterial cylinder, in which the transformation relation and the material parameters of the metamaterial are obtained based on transformation optics. Numerical simulations confirm the theoretical predictions and further demonstrate the ability of such a new scheme to generate arbitrary OAM of photons. The result provides a new route to manipulate the phase of electromagnetic fields and may be valuable in developing photonic devices based on OAMs.

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Phase is an important characteristics of electromagnetic waves. As is well known, the beam with a helical phase, typically Laguerre-Gaussian (LG) beam, has an azimuthal wavefront dependence of $\exp(-il\phi)$ and carries orbital angular momentum (OAM) $l\hbar$ [1, 2]. Due to its exotic properties, such a beam has attracted considerable interest and has been shown to have valuable applications in manipulating microparticles [3], optical communications [4], quantum information [5], optical data storage [6], as well as in biophysics [7]. To generate helical beams, mode coupling in laser cavity, computer holograms and spiral phase plates are often used, though the mode impurity and the limited beam intensity are involved [1, 2].

It is always desirable to have full control of the phase. For this purpose, recently intensively studied metamaterials provide a promising route. Metamaterials designed on the basis of transformation optics [8], also called transformation media, can be used in “controlling electromagnetic fields”, thus leads to unique potential applications in photonic devices [9–16]. To control the phase, recent work included the conversion of cylindrical waves into plane waves [17] and the rotation of the phase [18, 19]. However, the phase was only changed inside the medium, but still planar outside. Then, a question arises naturally: whether the output wave can be endowed with phase desired, such as a helicoidal wavefront, i.e. LG beams?

However, the problem whether it is possible to generate OAM by metamaterials has been unclear until now. We believe this problem is important to use the unconventional properties of OAM in metamaterials found most recently and to control angular momenta for further application in photonic devices [20, 21]. We also believe this problem can be solved by metamaterials since nanoscale structures have been utilized to convert spin angular momenta into OAMs for surface electromagnetic waves [22, 23].

In this work, therefore, we demonstrate theoretically

and numerically how to use metamaterials to form a helical wavefront for a beam and thus to enable the photon to carry orbital angular momenta (OAMs). Specifically, we will show the transformation of a Gaussian beam into a LG beam [1, 24] by a metamaterial cylinder. Such a metamaterial cylinder can be theoretically regarded reflectionless [25, 26], so the conversion efficiency can be high. What’s more, the adaptive behavior of reconfigurable metamaterials enables convenient generation of arbitrary OAMs of photons. In addition, since the figuration is cylinder and the units of metamaterials can support high power beams, the cylinder might avoid the above mentioned drawbacks troubled with the conventional methods to generate helical beams. Our result provides a new approach to control the phase of electromagnetic fields and to manipulate the Hall effect of light (HEL) [27, 28] and may be valuable in utilizing the HEL and developing photonic devices based on OAMs.

For a general LG_p^l beam [1, 20, 24], the field is $E_{pl}(r, \phi, x) = u_{pl}e^{-ikx}$ with

$$u_{pl}(r, \phi, x) = \frac{C_{pl}}{w(x)} \left[\frac{\sqrt{2}r}{w(x)} \right]^{|l|} L_p^{|l|} \left(\frac{2r^2}{w^2(x)} \right) \exp \left[\frac{-r^2}{w^2(x)} - \frac{ikr^2}{2R(x)} + i(2p + l + 1) \tan^{-1} \frac{x}{x_R} - il\phi \right], \quad (1)$$

where $w(x) = w_0 \sqrt{1 + (x/x_R)^2}$ is the beam size, $R(x) = x + x_R^2/x$ is the radius of curvature of the wave front, C_{pl} is the normalization constant, L_p^l is a generalized Laguerre polynomial, $x_R = kw_0^2/2$ is the Rayleigh length. Obviously, the wavefront is helical, as illustrated in Fig. 1. Correspondingly, the momentum density is [1]

$$\mathbf{P}(r, \phi, x) = \sqrt{\epsilon_0 \mu_0} \left[\frac{rx}{x^2 + x_R^2} |u|^2 \hat{r} + \frac{l}{kr} |u|^2 \hat{\phi} + |u|^2 \hat{x} \right]. \quad (2)$$

Obviously, the momentum of photon has an azimuthal component that is l/kr times the x component. If the linear momentum of each photon is given by $\hbar k$, then the OAM per photon is $l\hbar$ by taking the cross-product of its azimuthal component with the radius vector \mathbf{r} . When $p = 0$ and $l = 0$, the LG beam degenerates into a Gaussian beam [1, 24]. Therefore, to generate OAM, it is

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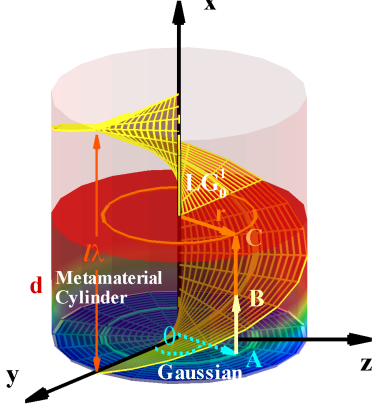


FIG. 1: A metamaterial cylinder that transforms Gaussian beam into a helical LG beam. The green spiral line denotes a trajectory of Poynting vector and the helicoid is the wavefront of the helical LG beam.

key to produce a helical wavefront. While in standard approach of transformation optics, the transformation is based on operation of light ray. So, we can obtain equal effect in manipulating electromagnetic field by operation on the phase. This is the foundation for our theoretical scheme to generate beams with OAMs. For simplicity, we consider how to convert a Gaussian beam into a LG_0^l beam, i.e., to generate a helical wavefront $\exp(-il\phi)$.

In the following, the key issue is to find the transformation relation, i.e. operation on the light phase. Following the basic approach of transformation optics [8, 9], the Jacobian transformation matrix between the transformed coordinate and the original coordinate is

$$\Lambda_{\alpha}^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^{\alpha}}. \quad (3)$$

The associated permittivity and permeability tensors of the transformed medium are

$$\begin{aligned} \varepsilon^{ij} &= |\det(\Lambda_i^{i'})|^{-1} \Lambda_i^{i'} \Lambda_j^{j'} \varepsilon^{ij}, \\ \mu^{ij} &= |\det(\Lambda_i^{i'})|^{-1} \Lambda_i^{i'} \Lambda_j^{j'} \mu^{ij}. \end{aligned} \quad (4)$$

Consider the circularly cylindrical device of length d and radius r_0 . At the same time, we set $d = l\lambda/n$ to ensure the helical phase l intertwined [1, 2], where λ is the wavelength in vacuum and n is a proportional coefficient,

$$n = l\lambda/d. \quad (5)$$

For a beam field propagating along the symmetrical axis x , it is desired that the wave front be helical in the $+\hat{\theta}$ by an angle 2π at the exit boundary $x = d$. This can be achieved by extruding the coordinate in the $+\hat{x}$ direction, $A \rightarrow B \rightarrow C$ as shown in Fig. 1. One such a transformation from the original circular cylindrical system (x, r, θ) to the transformed one (x', r', θ') is given by the mapping

$$x' = 2\pi x/n\theta, r' = r, \theta' = \theta. \quad (6)$$

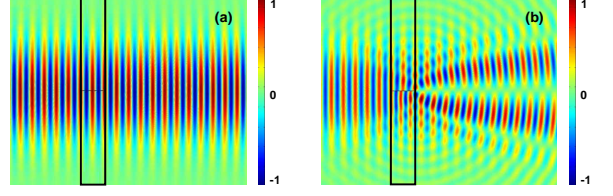


FIG. 2: Normalized distributions of the field magnitude through (a) vacuum and (b) transformation medium plate in the longitudinal section.

Then the Jacobian transformation matrix Eq. (3) becomes

$$\Lambda_{\alpha}^{\alpha'} = \begin{bmatrix} 2\pi/n\theta' & 0 & -x'/\theta'r' \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (7)$$

with $\det(\Lambda_{\alpha}^{\alpha'}) = 2\pi/(n\theta')$. Note that the transformation is that the original system is squeezed/expanded in the \hat{x} direction as a function of increasing θ , in contrast with that in controlling the polarization where the original cylindrical volume is twisted in the $\hat{\theta}$ direction as x increases [16]. By the above transformation, we obtain the permittivity and permeability tensors of the cylinder as

$$\varepsilon = \mu = \begin{bmatrix} \frac{2\pi}{n\theta'} + \frac{nx'^2}{2\pi\theta'r'^2} & 0 & -\frac{nx'}{2\pi r'} \\ 0 & \frac{n\theta'}{2\pi} & 0 \\ -\frac{nx'}{2\pi r'} & 0 & \frac{n\theta'}{2\pi} \end{bmatrix}. \quad (8)$$

Because the coordinate expansion or compression only occurs in x direction, the transformation cylinder can be regarded reflectionless [25] and then the conversion efficiency can be high.

For convenience of application we rewrite the above results in the Cartesian coordinate $Oxyz$. The transformation is

$$x' = 2\pi x/(n\theta + c), y' = y, z' = z, \quad (9)$$

where $\theta = \tan^{-1}(z/y)$ is the azimuthal angle in the Oyz plane and c related to the initial point coordinate is introduced to avoid the singularity in calculation. Note that c would not change the result of OAM though may affect the field. The Jacobian transformation matrix Eq. (3) is

$$\Lambda_{\alpha}^{\alpha'} = \begin{bmatrix} \frac{2\pi}{(n\theta+2c\pi)} & \frac{nx'z'}{(n\theta'+2c\pi)r'^2} & -\frac{nx'y'}{(n\theta+2c\pi)r'^2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (10)$$

with $\det(\Lambda_{\alpha}^{\alpha'}) = 2\pi/(n\theta + 2c\pi)$, where $r = \sqrt{y^2 + z^2}$. The permittivity and permeability tensors of the cylinder are respectively

$$\varepsilon = \mu = \begin{bmatrix} \frac{2\pi}{n\theta+2c\pi} + \frac{n^2x'^2}{2\pi(n\theta+2c\pi)r'^2} & \frac{nx'z'}{2\pi r'^2} & -\frac{nx'y'}{2\pi r'^2} \\ \frac{nx'z'}{2\pi r'^2} & \frac{n\theta+2c\pi}{2\pi} & 0 \\ -\frac{nx'y'}{2\pi r'^2} & 0 & \frac{n\theta+2c\pi}{2\pi} \end{bmatrix} \quad (11)$$

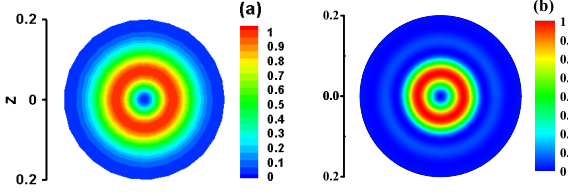


FIG. 3: Normalized transverse intensities of the transmitted beam at the plane 3λ away from the exit surface. (a) is the theoretical result and (b) is the simulated one.

From the above, we know that the wavefront transformation is $nd\theta/2\pi$ and the longitudinal phase shift is $nd\theta/\lambda$ at the position with azimuth angle θ . Recalling the knowledge of OAM [1, 2, 24], the OAM in the \hat{x} direction is

$$L = \frac{nd}{\lambda} \hbar = l\hbar. \quad (12)$$

The physics behind the generation of OAM can be understood simply. From Eq. (8) we see that the transverse material parameters equal to $n\theta/2\pi$, so the refraction index can be shown to be helically distributed and then the same occurs to the ray trajectory. Note that only the diagonal elements affect the behavior of transverse polarized waves determinatively. As a result, the phase difference is $l\lambda\theta/2\pi$ and then the helical wave front is $\exp(-il\phi)$. This further proves that the above theoretical result, Eq. (8) or (11), is correct.

In order to confirm the theoretical results, we make full-wave simulations based on the method of finite elements. We choose a transverse electric Gaussian beam with the waist $w_0 = 0.1m$ the frequency 6 GHz normally incident on a cylinder of $d = 0.1m$ and $r_0 = 0.2m$. To clearly show the field in the cylinder, we take $c = 1$. The distribution of the field magnitude through a vacuum and a transformation medium plate in the longitudinal section. are illustrated in Figs. 2(a) and (b). Fig. 2(b) shows that the field is zero along the beam axis which indicates a phase singularity, a typical characteristic of helical beams [2]. It shows that the phase of the lower half section of the cylinder is π larger than that of the upper half section. It should be noted that cd/λ gives the cycle of wave in the cylinder at the $\theta = 0$ section. Although it may affect the field illustrated in the two figures, however, c would not change l or the OAM.

To further validate the result, we show the intensity on the transverse section at the exit surface for the theoretical result by Eq. (2) in (a) and the simulated result in (b) of Fig. 3. In that figure, we can see that the intensity is a ring, which is a typical distribution of LG beam [2]. In Fig. 3(b), the intensity maximum is smaller than the analytical result because there exist scattering and diffraction as can be seen in Fig. 2. At the same time, in order to vividly demonstrate the helical evolvement of phase, we illustrate the phase distribution in Fig. 4. This figure shows the simulated result agrees satisfactorily with

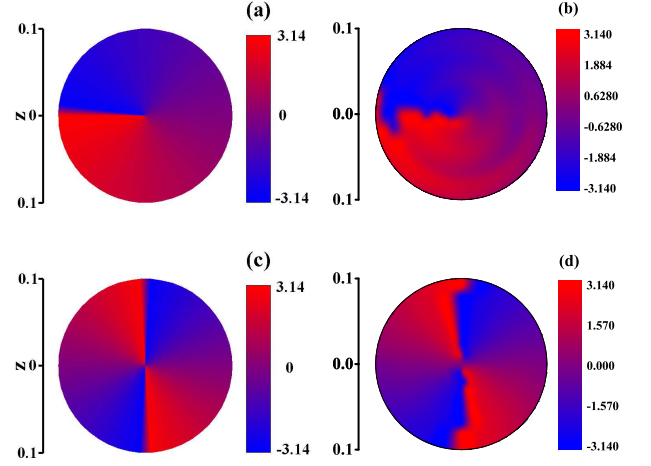


FIG. 4: Phase profiles of the transmitted beam for $l = 1$ for (a) and (b) and $l = 2$ for (c) and (d) within the range of one waist at the exit surface. (a) and (c) is the theoretical results, while (b) and (d) is the simulated ones. Note that the normal to the picture plane is contrary to the propagation direction \hat{x} . The result in (d) is obtained by overlapped two pieces of the unit OAM generator in (b).

the theoretical one by Eq. (2) in that the phase changes gradually from $-\pi$ to π . That immediately proves that the wavefront of the resultant beam is $l = 1$ intertwined [Figs. 4 (a) and (b)]. In terms of the basic knowledge of helical beams [1, 2], we conclude the photon has possessed an OAM of $L = 1 \cdot \hbar$.

A scheme can not be of any interest if the means of realizing it are not available. Fortunately several recent developments make such an OAM generator practically possible. Reviewing past work, we know that metamaterials with material parameters connected with the radius have been constructed in realizing electromagnetic cloaks in Ref. [9]. Those with material parameters related with the angle have been realized recently by Chen *et al* in rotating the field phase in Ref. [18]. Following their methods, hence, there essentially does not exist any technological obstacle to design the parameters of Eq. (8) which are related with both radial and azimuthal positions to construct the OAM generator. For example, we can use Chen's units, but place them helically in the \hat{x} direction. Since we only put the helical distribution in the transverse section in Ref. [18] into the longitudinal direction, we believe the scheme is realizable. On the other hand, there exists much easier way to generate OAM. We find that only the diagonal elements in Eq. (8) determine the OAM and the effect of elements related to longitudinal coordinate can be omitted. Therefore, one can generate OAM more easily by more simplified parameters, e.g. $\epsilon(\mu) = \text{In}\theta'/2\pi$ and $\mu(\epsilon) = \text{diag}[2\pi/n\theta', n\theta'/2\pi, n\theta'/2\pi]$ for TE(TM) waves.

Before ending this work, it is valuable to make some remarks. Firstly, by the above method, i.e. Eq. (8) or (11), one can realize tunable generator of OAM by changing

the parameters of metamaterials. Actually, the simplest method is to superpose the above generator of unit OAM to obtain any integer OAMs. An example is shown in Figs. 4 (c) and (d), where two pieces of the unit OAM generator are overlapped to generate OAM $l = 2$. Secondly, the present scheme realize the phase desired out of the material, different from the past work [18] because the finite embedded coordinate method [25, 26] have been applied here. Thirdly, the OAM does not be generated from nihility, but involves exchange of momenta of the beam and the cylinder, while the total OAMs are conserved. Usually this leads to mechanical consequences which can be measured [1, 2].

In conclusion, we have proposed to use metamaterials to enable a beam to carry OAM based on transformation optics. By transforming the planar wavefront into a helioidal one, we obtain the space transformation and work out the material parameters. Further we make numerical simulations and verify the correctness and feasibility of the theoretical result. We find that by superposing particular pieces of the unit OAM generator one can obtain

any integer OAMs. Moreover, if made of tunable and reconfigurable metamaterials, the cylinder would allow generating arbitrary OAMs. We also schematically point out how to construct such an OAM generator by the recent development in metamaterials. Our result provides a new route to accurately control the phase of electromagnetic fields and to manipulate the orbital HEL and may be valuable in developing photonic devices based on OAMs. Further work is desired to experimentally realize the theoretical scheme. In addition, it is much meaningful to investigate how to use metamaterials to generate arbitrary angular momentum in beams.

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- [1] L. Allen, M. W. Beijersbergen, R. J. C. Spreeuw and J. P. Woerdman, *Phys. Rev. A* **45**, 8185 (1992).
 - [2] L. Allen, S. M. Barnett, and M. J. Padgett, *Optical Angular Momentum*, (Institute of Physics Publishing, Bristol and Philadelphia, 2003); M. Padgett, J. Courtial, and L. Allen, *Physics Today* **57**, 35 (2004).
 - [3] N. B. Simpson, K. Dholakia, L. Allen, and M. J. Padgett, *Opt. Lett.* **22**, 52 (1997).
 - [4] G. Gibson, J. Courtial, M. J. Padgett, M. Vasnetsov, V. Pas'ko, S. M. Barnett, S. Franke-Arnold, *Opt. Express* **12**, 5448 (2004).
 - [5] A. Mair, A. Vaziri, G. Weihs, and A. Zeilinger, *Nature* **412**, 313 (2001); G. Molina-Terriza, J. P. Torres, and L. Torner, *Nature Phys.* **3**, 305 (2007).
 - [6] R. J. Voogd, M. Singh, S. Pereira, A. van de Nes, and J. Braat, in *Frontiers in Optics*, OSA Technical Digest Series, paper FTuG14 (2004).
 - [7] D. G. Grier, *Nature* **424**, 810 (2003).
 - [8] J. B. Pendry, D. Schurig, and D. R. Smith, *Science* **312**, 1780 (2006).
 - [9] D. Schurig, J. J. Mock, B. J. Justice, S. A. Cummer, J. B. Pendry, A. F. Starr, and D. R. Smith, *Science* **314**, 977 (2006); W. Cai, U. K. Chettiar, A. V. Kildishev, and V. M. Shalaev, *Nature Photon.* **1**, 224 (2007).
 - [10] M. Rahm, D. Schurig, D. A. Roberts, S. A. Cummer, D. R. Smith, and J. B. Pendry, *Photonics Nanostructures* **6**, 87 (2008); M. Rahm, D. A. Roberts, J. B. Pendry, and D. R. Smith, *Opt. Express* **16**, 11555 (2008).
 - [11] X. F. Xu, Y. J. Feng and T. Jiang, *New J. Phys.* **10**, 115027 (2008).
 - [12] J. Allen, N. Kundtz, D. A. Roberts, S. A. Cummer, and D. R. Smith, *Appl. Phys. Lett.* **94**, 194101 (2009).
 - [13] N. I. Landy and W. J. Padilla, *Opt. Express* **17**, 14872 (2009).
 - [14] E. E. Narimanova and A. V. Kildisheva, *Appl. Phys. Lett.* **95**, 041106 (2009).
 - [15] M. A. Moiseeva and L. L. Doskolovich, *J. Mod. Opt.* **57**, 536 (2010).
 - [16] D. H. Kwon and D. H. Werner, *Opt. Express* **16**, 18731 (2008).
 - [17] W. X. Jiang, J. Y. Chin and T. J. Cui, *Materials Today* **12**, 26 (2009).
 - [18] H. Y. Chen, B. Hou, S. Y. Chen, X. Y. Ao, W. J. Wen, and C. T. Chan, *Phys. Rev. Lett.* **102**, 183903 (2009).
 - [19] Y. Luo, H. Chen, J. Zhang, L. Ran, and J. A. Kong, *Phys. Rev. B* **77**, 125127 (2008).
 - [20] H. Luo, Z. Ren, W. Shu, and S. Wen, *Phys. Rev. A* **77**, 023812 (2008); H. Luo, S. Wen, W. Shu, Z. Tang, Y. Zou, and D. Fan, *Phys. Rev. A* **80**, 043810 (2009).
 - [21] A. Aiello, C. Marquardt, and G. Leuchs, *Phys. Rev. A* **81**, 053838 (2010).
 - [22] Y. Gorodetski, N. Shitrit, I. Bretner, V. Kleiner, and E. Hasman, *Nano. Lett.* **9**, 3016 (2009).
 - [23] L. T. Vuong, A. J. L. Adam, J. M. Brok, P. C. M. Planken, and H. P. Urbach, *Phys. Rev. Lett.* **104**, 083903 (2010).
 - [24] A. V. Volyar, V. G. Shvedov, and T. A. Fadeeva, *Tech. Phys. Lett.* **25**, 203 (1999).
 - [25] W. Yan, M. Yan, M. Qiu, arXiv: 0806.3231, (2008).
 - [26] M. Rahm, S. A. Cummer, D. Schurig, J. B. Pendry, and D. R. Smith, *Phys. Rev. Lett.* **100**, 063903 (2008).
 - [27] M. Onoda, S. Murakami, and N. Nagaosa, *Phys. Rev. Lett.* **93**, 083901 (2004).
 - [28] K. Y. Bliokh, A. Niv, V. Kleiner, and E. Hasman, *Nature Photonics* **2**, 748 (2008); K. Y. Bliokh, *Phys. Rev. Lett.* **97**, 043901 (2006); K. Y. Bliokh, Y. Gorodetski, V. Kleiner, and E. Hasman, *Phys. Rev. Lett.* **101**, 030404 (2008).